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Hyperbolic propagation of an axisymmetric thermal signal in an infinite solid medium

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Abstract—Thermal wave propagation in an infinite solid medium which surrounds an infinitely long cylindrical surface is considered. This surface transfers a prescribed and time-dependent heat flux to the solid medium. The non-stationary heat conduction problem is studied by assuming a non-vanishing value of the thermal relaxation time for the solid medium, i.e. by employing the hyperbolic heat conduction equation. An analytical expression of the temperature field in the solid is determined. Examples are provided for heat fluxes which vary with time as a square wave pulse or as a triangular wave pulse. Comparisons with the solutions obtained for parabolic heat conduction are performed. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

Ever since Maxwell's paper [1], it is widely accepted in the literature that only for stationary or weekly non-stationary temperature fields the constitutive equation which relates the heat flux density vector $\mathbf{q}(\mathbf{x}, t)$ to the temperature gradient $\nabla T(\mathbf{x}, t)$ is given by Fourier's law

$$\mathbf{q}(\mathbf{x}, t) = -k\nabla T(\mathbf{x}, t). \tag{1}$$

If the time scale of local temperature variations is very small, equation (1) is replaced by

$$\mathbf{q}(\mathbf{x}, t+\tau) = -k\nabla T(\mathbf{x}, t) \tag{2}$$

where τ, called thermal relaxation time, is a thermodynamic property of the material where heat conduction occurs. Unlike Fourier's law, the constitutive equation (2) is non-local in time. However, the local character of the heat conduction theory is usually restored by a truncation of equation (2) to the first order in τ, namely

$$\mathbf{q}(\mathbf{x}, t)+\tau\frac{\partial \mathbf{q}(\mathbf{x}, t)}{\partial t} = -k\nabla T(\mathbf{x}, t). \tag{3}$$

If heat generation is present within the material, the local energy balance can be expressed as

$$-\nabla\cdot\mathbf{q}(\mathbf{x}, t)+q_g(\mathbf{x}, t) = \frac{k}{\alpha}\frac{\partial T(\mathbf{x}, t)}{\partial t} \tag{4}$$

where $q_g(\mathbf{x}, t)$ is the power generated per unit volume and α is the thermal diffusivity. Equations (3) and (4) yield the hyperbolic heat conduction equation, i.e.

$$k\nabla^2 T(\mathbf{x}, t)+q_g(\mathbf{x}, t)+\tau\frac{\partial q_g(\mathbf{x}, t)}{\partial t} = \frac{k}{\alpha}\left[\frac{\partial T(\mathbf{x}, t)}{\partial t}+\tau\frac{\partial^2 T(\mathbf{x}, t)}{\partial t^2}\right]. \tag{5}$$

The constitutive equation (3) predicts a finite speed for the propagation of thermal signals with a value $\sqrt{(\alpha/\tau)}$. A review of the physical bases of hyperbolic heat conduction can be found in [2], while more recent theoretical and experimental results obtained in this field are reviewed by Özisik and Tzou [3].

Most of the experimental studies on the phenomenon of finite-speed propagation of thermal signals, often called *second sound*, have been performed at low temperatures. For instance, in [4] this phenomenon has been observed in NaF at ≈ 10 K, while in [5] it has been shown that the speed of second sound in Bi at ≈ 3.4 K is 780 m s^{-1}. On the other hand, Kaminski [6] has performed measurements of the thermal relaxation time at room temperature on some non-homogeneous materials such as sand or glass ballotini. Kaminski has shown that the values of the thermal relaxation time for these materials lie in the range 10 s $< \tau <$ 50 s. Further experimental validations of the hyperbolic heat conduction equation could be based on the comparison between solutions of the equation and measurements of the temperature field performed by suitable experimental apparatuses. Then, values of the thermal relaxation time or of the speed of propagation could be obtained by a parameter estimation method.

In the literature, many solutions of the hyperbolic heat conduction equation have been determined. Most of these solutions refer to propagation of thermal waves either in semi-infinite solid media bounded by a plane surface or in infinite plane slabs. For instance, Baumeister and Hamill [7], Vick and Özisik [8], Glass *et al.* [9], Orlande and Özisik [10] have

NOMENCLATURE

a	dimensionless function of η and ξ defined by equation (26)
$c_1(s)$, $c_2(s)$	integration constants employed in equation (18)
f	dimensionless function of η and ξ defined by equation (24)
$F(t)$	$=q(t)/q_0$, dimensionless function of time
g	dimensionless function of ξ defined by equation (13)
H	Heaviside's unit step function
i	$=\sqrt{-1}$, imaginary unit
Im	imaginary part of a complex number
I_n	modified Bessel function of first kind and order n
J_n	Bessel function of first kind and order n
k	thermal conductivity
K_n	modified Bessel function of second kind and order n
L	Laplace transform operator
n	integer number
$q(t)$	heat flux which crosses the surface $r=r_0$
$\mathbf{q}$	heat flux density vector
q_g	power generated per unit volume
q_0	constant heat flux
r	radial coordinate
R	radius of the semicircular path Γ_R
Re	real part of a complex number
r_0	radius of the heating surface
s	Laplace transformed variable
t	time
T	temperature
T_0	initial temperature
u	$=\sqrt{y}$, integration variable
$\mathbf{x}$	position vector
y	integration variable
Y_n	Bessel function of second kind and order n
z	complex variable.

Greek symbols

α	thermal diffusivity
γ	constant employed in the inversion formula (28)
Γ_R, Γ_{ε_1}, Γ_{ε_2}	paths represented in Fig. 1
δ	Dirac's delta distribution
ε_1, ε_2	radiuses of the small circles Γ_{ε_1} and Γ_{ε_2}
η	$=r/r_0$, dimensionless radial coordinate
ϑ	dimensionless temperature defined by equation (9)
λ	integration variable
ξ	$=\alpha t/r_0^2$, dimensionless time
ξ_0	dimensionless value of the pulse switching-on time
ξ_1	dimensionless value of the pulse switching-off time
Ξ	$=\alpha\tau/r_0^2$, dimensionless parameter
ρ	dimensionless function defined by equation (30)
Σ	closed path represented in Fig. 1
τ	thermal relaxation time
φ	dimensionless function of ξ defined by equation (35)
$\phi(\xi)$	$=F(r_0^2\xi/\alpha)$, dimensionless function of ξ
χ	dimensionless function of ξ employed in equation (33)
ψ	dimensionless function of y, η and ξ defined by equation (32).

Superscripts

$\bar{}$	Laplace transformed function.

found solutions of the hyperbolic heat conduction equation in a semi-infinite solid bounded by a plane surface, with different boundary conditions. In particular, in ref. [7] a step change of the temperature of the boundary surface is considered, while in refs [8–10] time-dependent heat fluxes are prescribed at this surface. On the other hand, Özisik and Vick [11], Frankel *et al.* [12] and Hector *et al.* [13] study hyperbolic propagation of thermal signals in an infinite plane slab. In [11], an internal heat generation is considered within the slab, while the surfaces are supposed to be insulated. In [12] and [13], no heat generation occurs within the slab and one of the surfaces is insulated. On the other surface, a time-dependent heat flux is prescribed, which is either uniform [12], or non-uniform and axisymmetric [13].

The aim of this paper is to study hyperbolic heat conduction in an infinite solid medium bounded internally by a circular cylindrical surface with radius r_0, with no heat generation in the solid and a prescribed time-dependent heat flux on the boundary surface. The choice of this system for the study of thermal wave propagation is motivated by the lack of serious difficulties in the setup of an experimental apparatus which reproduces it. Indeed, the cylindrical geometry of the heating surface is infinite only in one direction and can be practically implemented by a sufficiently long cylindrical electric resistor. Provided that the heat capacity of the resistor is very small, any heat flux signal can be experimentally reproduced by a suitable non-stationary electric current.

The paper is organized as follows. The system under examination is described and the governing equations are written. Then, an analytical solution of the problem is determined for an arbitrary time-dependent heat flux at the boundary surface. Finally, examples are provided in which the general solution is applied to heat fluxes which vary with time like a square wave or a triangular wave pulse. Comparisons are made in these cases between parabolic and hyperbolic heat conduction.

MATHEMATICAL MODEL

In this section, the hyperbolic heat conduction equation for the axisymmetric problem under exam is presented together with its initial and boundary conditions. Then, both the equation and the initial and boundary conditions are written in the Laplace transform domain.

Let us consider an infinitely long cylindrical surface with radius r_0 which internally bounds a homogeneous solid which occupies the region $r_0 < r < +\infty$. It will be assumed that the thermal conductivity k, the thermal diffusivity α and the thermal relaxation time τ of the solid can be treated as constants. No heat generation is supposed to be present within the solid, so that $q_g(\mathbf{x}, t) = 0$. Moreover, at the initial time $t = 0$ the temperature field within the solid is uniform with a value T_0 and stationary, while for $t > 0$ a uniform and time-dependent heat flux $q(t) = q_0F(t)$ crosses radially the surface $r = r_0$, where $F(t)$ is a dimensionless function of time. Therefore, the temperature field in the solid is axisymmetric, and equation (5) can be rewritten as

$$\alpha\left[\frac{\partial^2 T(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial T(r,t)}{\partial r}\right]=\frac{\partial T(r,t)}{\partial t}+\tau\frac{\partial^2 T(r,t)}{\partial t^2} \tag{6}$$

where the initial and boundary conditions are given by

$$T(r,0) = T_0 \quad \left.\frac{\partial T}{\partial t}\right|_{t=0} = 0 \tag{7}$$

$$-k\left.\frac{\partial T}{\partial r}\right|_{r=r_0} = q_0\left[F(t)+\tau\frac{\mathrm{d}F(t)}{\mathrm{d}t}\right], \quad t>0. \tag{8}$$

By introducing the dimensionless radius $\eta = r/r_0$, the dimensionless time $\xi = \alpha t/r_0^2$, the dimensionless parameter $\Xi = \alpha\tau/r_0^2$, and the dimensionless temperature

$$\vartheta(\eta,\xi) = k\frac{T(r,t)-T_0}{q_0 r_0} \tag{9}$$

equations (6)–(8) can be expressed in a dimensionless form, i.e.

$$\frac{\partial^2\vartheta(\eta,\xi)}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial\vartheta(\eta,\xi)}{\partial\eta}=\frac{\partial\vartheta(\eta,\xi)}{\partial\xi}+\Xi\frac{\partial^2\vartheta(\eta,\xi)}{\partial\xi^2} \tag{10}$$

$$\vartheta(\eta,0) = 0 \quad \left.\frac{\partial\vartheta}{\partial\xi}\right|_{\xi=0} = 0 \tag{11}$$

$$\left.\frac{\partial\vartheta}{\partial\eta}\right|_{\eta=1} = -\phi(\xi)-\Xi\frac{\mathrm{d}\phi(\xi)}{\mathrm{d}\xi}, \quad \xi>0 \tag{12}$$

where $\phi(\xi) = F(r_0^2\xi/\alpha)$. Let function $g(\xi)$ be defined as

$$g(\xi) = \phi(\xi)+\Xi\frac{\mathrm{d}\phi(\xi)}{\mathrm{d}\xi}. \tag{13}$$

Then, equation (12) yields

$$\left.\frac{\partial\vartheta}{\partial\eta}\right|_{\eta=1} = -g(\xi), \quad \xi>0. \tag{14}$$

By employing the Laplace transform of $\vartheta(\eta,\xi)$ with respect to ξ,

$$\tilde{\vartheta}(\eta,s) = \int_0^{+\infty} \mathrm{e}^{-s\xi}\,\vartheta(\eta,\xi)\,\mathrm{d}\xi \tag{15}$$

and the initial conditions (11), equations (10) and (14) can be rewritten as

$$\frac{\partial^2\tilde{\vartheta}(\eta,s)}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial\tilde{\vartheta}(\eta,s)}{\partial\eta}-(s+\Xi s^2)\tilde{\vartheta}(\eta,s) = 0 \tag{16}$$

$$\left.\frac{\partial\tilde{\vartheta}}{\partial\eta}\right|_{\eta=1} = -\tilde{g}(s) \tag{17}$$

where $\tilde{g}(s)$ is the Laplace transform of $g(\xi)$.

EVALUATION OF THE TEMPERATURE FIELD

In this section, equations (16) and (17) are solved. Then, an analytical expression of the inverse Laplace transform of $\tilde{\vartheta}(\eta,s)$ is obtained.

Equation (16) is a Bessel-type equation and its general solution can be expressed as [14]

$$\tilde{\vartheta}(\eta,s) = c_1(s)I_0(\eta\sqrt{(s+\Xi s^2)}) + c_2(s)K_0(\eta\sqrt{(s+\Xi s^2)}) \tag{18}$$

where $c_1(s)$ and $c_2(s)$ are arbitrary functions of s. Since the temperature field $T(r,t)$ for $r\to+\infty$ must be equal to its initial value T_0 for every time $t \geqslant 0$, both $\vartheta(\eta,\xi)$ and its Laplace transform $\tilde{\vartheta}(\eta,s)$ must tend to zero for $\eta\to+\infty$. Therefore, by recalling the asymptotic properties of Bessel functions [14]

$$\lim_{z\to+\infty} I_0(z) = +\infty \tag{19}$$

$$\lim_{z\to+\infty} K_0(z) = 0 \tag{20}$$

where z is a complex variable, and by employing equation (18), it can be concluded that $c_1(s)$ must be identically zero. On the other hand, $c_2(s)$ can be determined by employing equation (17). In fact, on account of the identity [14]

$$\frac{\mathrm{d}K_0(z)}{\mathrm{d}z} = -K_1(z) \tag{21}$$

equations (17) and (18) yield

$$c_2(s) = \frac{\tilde{g}(s)}{K_1(\sqrt{(s+\Xi s^2)})\sqrt{(s+\Xi s^2)}}. \tag{22}$$

Therefore, the solution of equations (16) and (17) is given by

$$\tilde{\vartheta}(\eta, s) = \tilde{g}(s)\frac{K_0(\eta\sqrt{(s+\Xi s^2)})}{K_1(\sqrt{(s+\Xi s^2)})\sqrt{(s+\Xi s^2)}}. \tag{23}$$

Let function $f(\eta, \xi)$ be defined as

$$f(\eta, \xi) = L^{-1}\left\{\frac{K_0(\eta\sqrt{(s+\Xi s^2)})}{K_1(\sqrt{(s+\Xi s^2)})\sqrt{(s+\Xi s^2)}}\right\} \tag{24}$$

where L^{-1} is the inverse Laplace transform operator. By definition, $f(\eta, \xi)$ does not depend on the time-evolution of the heat flux prescribed at $r = r_0$. On account of the convolution theorem for Laplace transforms [15], equations (23) and (24) yield

$$\vartheta(\eta, \xi) = \int_0^{\xi} f(\eta, \lambda) g(\xi-\lambda)\,\mathrm{d}\lambda. \tag{25}$$

Moreover, let function $a(\eta, \xi)$ be defined as

$$a(\eta, \xi) = L^{-1}\left\{\frac{K_0(\eta\sqrt{(s+\Xi s^2)})\,\mathrm{e}^{(\eta-1)s\sqrt{\Xi}}}{K_1(\sqrt{(s+\Xi s^2)})\sqrt{(s+\Xi s^2)}}\right\}. \tag{26}$$

As a consequence of the translation property of the inverse Laplace transform [15], equations (24) and (26) yield

$$f(\eta, \xi) = H(\xi-(\eta-1)\sqrt{\Xi})\,a(\eta, \xi-(\eta-1)\sqrt{\Xi}) \tag{27}$$

where H is Heaviside's unit step function.
The right-hand side of equation (26) can be evaluated by the inversion formula for Laplace transforms

$$a(\eta, \xi) = \frac{1}{2\pi \mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{s\xi}\frac{K_0(\eta\sqrt{(s+\Xi s^2)})\,\mathrm{e}^{(\eta-1)s\sqrt{\Xi}}}{K_1(\sqrt{(s+\Xi s^2)})\sqrt{(s+\Xi s^2)}}\,\mathrm{d}s \tag{28}$$

where γ is a real number greater than the real part of any singularity of the integrand [15]. The evaluation of the integral at the right-hand side of equation (28) is performed in the Appendix by a contour integration which employs the closed path represented in Fig. 1. As a consequence of the contour integration, equation (28) can be written as

$$a(\eta, \xi) = \frac{1}{\pi}\int_0^{1/\Xi}\frac{J_1(\rho(y))Y_0(\rho(y)\eta) - Y_1(\rho(y))J_0(\rho(y)\eta)}{\rho(y)[J_1(\rho(y))^2 + Y_1(\rho(y))^2]}$$
$$\times\,\mathrm{e}^{-y[\xi+(\eta-1)\sqrt{\Xi}]}\,\mathrm{d}y$$
$$+\int_{1/\Xi}^{+\infty}\frac{I_1(\rho(y))K_0(\rho(y)\eta) + K_1(\rho(y))I_0(\rho(y)\eta)}{\rho(y)[\pi^2 I_1(\rho(y))^2 + K_1(\rho(y))^2]}$$
$$\times\,\mathrm{e}^{-y[\xi+(\eta-1)\sqrt{\Xi}]}\,\mathrm{d}y \tag{29}$$

where function $\rho(y)$ is defined as

$$\rho(y) = \sqrt{(y|1-\Xi y|)}. \tag{30}$$

On account of equations (25), (27) and (29), $\vartheta(\eta, \xi)$ is given by

$$\vartheta(\eta, \xi) =$$
$$\frac{1}{\pi}\int_0^{1/\Xi}\frac{J_1(\rho(y))Y_0(\rho(y)\eta) - Y_1(\rho(y))J_0(\rho(y)\eta)}{\rho(y)[J_1(\rho(y))^2 + Y_1(\rho(y))^2]}$$
$$\times\,\psi(y, \eta, \xi)\,\mathrm{d}y$$
$$+\int_{1/\Xi}^{+\infty}\frac{I_1(\rho(y))K_0(\rho(y)\eta) + K_1(\rho(y))I_0(\rho(y)\eta)}{\rho(y)[\pi^2 I_1(\rho(y))^2 + K_1(\rho(y))^2]}$$
$$\times\,\psi(y, \eta, \xi)\,\mathrm{d}y \tag{31}$$

where function $\psi(y, \eta, \xi)$ is defined as

$$\psi(y, \eta, \xi)$$
$$= H(\xi-(\eta-1)\sqrt{\Xi})\,\mathrm{e}^{-y\xi}\int_0^{\xi-(\eta-1)\sqrt{\Xi}} \mathrm{e}^{y\lambda} g(\lambda)\,\mathrm{d}\lambda. \tag{32}$$

For every time-evolution of the heat flux at $r = r_0$, i.e. for every function $\phi(\xi)$, equations (13), (31) and (32) allow the determination of the dimensionless temperature field within the solid.

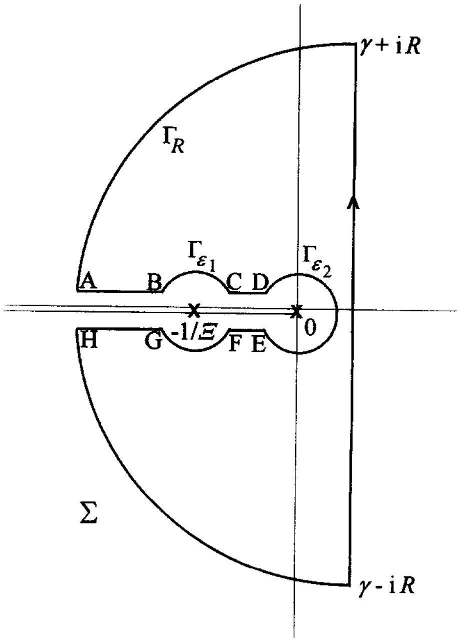

Fig. 1. Closed path employed in the evaluation of $a(\eta, \xi)$.

As a consequence of equations (31) and (32), the dimensionless temperature ϑ is zero for every value of η such that $\eta > 1+\xi/\sqrt{\Xi}$, i.e. the temperature equals the initial value T_0 at every position r such that $r > r_0+t\sqrt{(\alpha/\tau)}$. This result is in agreement with the physical interpretation of $\sqrt{(\alpha/\tau)}$ as the speed of propagation of thermal signals.

PROPAGATION OF SQUARE WAVE AND TRIANGULAR WAVE THERMAL SIGNALS

In this section, equations (13), (31) and (32) are employed in the case the heat flux at $r = r_0$ is nonzero only for a finite time interval and varies either as square wave pulse or as a triangular wave pulse.

If the heat flux at $r = r_0$ behaves as a pulse, function $\phi(\xi)$ can be expressed as

$$\phi(\xi) = [H(\xi-\xi_0)-H(\xi-\xi_1)]\chi(\xi) \tag{33}$$

where ξ_0 and ξ_1 are two positive real constants such that the pulse starts when $t = r_0^2\xi_0/\alpha$ and ends when $t = r_0^2\xi_1/\alpha$. Function $\chi(\xi)$ is arbitrary; in particular, $\chi(\xi) = 1$ for the square wave pulse represented in Fig. 2(a), while $\chi(\xi) = (\xi-\xi_0)/(\xi_1-\xi_0)$ for the triangular wave pulse represented in Fig. 2(b). On account of equations (13) and (33), function $g(\xi)$ is given by

$$g(\xi) = [H(\xi-\xi_0)-H(\xi-\xi_1)]\varphi(\xi) + \Xi\chi(\xi)\frac{d}{d\xi}[H(\xi-\xi_0)-H(\xi-\xi_1)] \tag{34}$$

where function $\varphi(\xi)$ is defined as

$$\varphi(\xi) = \chi(\xi)+\Xi\frac{d\chi(\xi)}{d\xi}. \tag{35}$$

As is well known [15], the derivative of Heaviside's unit step function H is Dirac's delta distribution δ, so that equation (34) can be rewritten as

$$g(\xi) = [H(\xi-\xi_0)-H(\xi-\xi_1)]\varphi(\xi) + \Xi\chi(\xi)[\delta(\xi-\xi_0)-\delta(\xi-\xi_1)]. \tag{36}$$

On account of equations (32) and (36), $\psi(y,\eta,\xi)$ can be expressed as

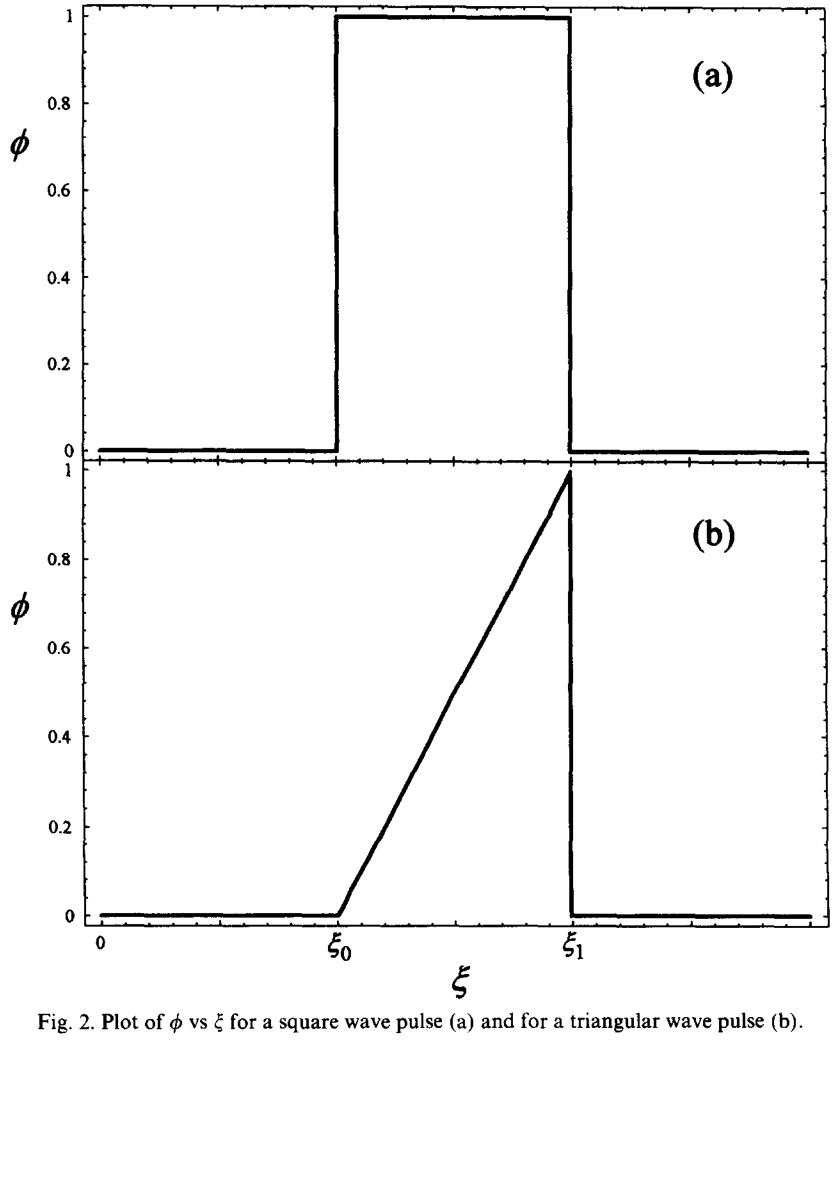

Fig. 2. Plot of ϕ vs ξ for a square wave pulse (a) and for a triangular wave pulse (b).

$$\psi(\gamma,\eta,\xi) = H(\xi-\xi_0-(\eta-1)\sqrt{\Xi})\,e^{-\gamma\xi}$$
$$\times\left[\Xi\chi(\xi_0)\,e^{\gamma\xi_0}+\int_{\xi_0}^{\xi-(\eta-1)\sqrt{\Xi}} e^{\gamma\lambda}\varphi(\lambda)\,d\lambda\right]$$
$$-H(\xi-\xi_1-(\eta-1)\sqrt{\Xi})\,e^{-\gamma\xi}$$
$$\times\left[\Xi\chi(\xi_1)\,e^{\gamma\xi_1}+\int_{\xi_1}^{\xi-(\eta-1)\sqrt{\Xi}} e^{\gamma\lambda}\varphi(\lambda)\,d\lambda\right]. \qquad (37)$$

If the heat flux at $r = r_0$ behaves as a square wave pulse, i.e. if $\chi(\xi) = 1$, equations (35) and (37) yield

$$\psi(\gamma,\eta,\xi) = H(\xi-\xi_0-(\eta-1)\sqrt{\Xi})$$
$$\times\left[\Xi\,e^{-\gamma(\xi-\xi_0)}+\frac{e^{-\gamma(\eta-1)\sqrt{\Xi}}-e^{-\gamma(\xi-\xi_0)}}{\gamma}\right]$$
$$-H(\xi-\xi_1-(\eta-1)\sqrt{\Xi})$$
$$\times\left[\Xi\,e^{-\gamma(\xi-\xi_1)}+\frac{e^{-\gamma(\eta-1)\sqrt{\Xi}}-e^{-\gamma(\xi-\xi_1)}}{\gamma}\right]. \qquad (38)$$

On the other hand, if the heat flux at $r = r_0$ behaves as a triangular wave pulse, i.e. if $\chi(\xi) = (\xi-\xi_0)/(\xi_1-\xi_0)$, equations (35) and (37) yield

$$\psi(\gamma,\eta,\xi) = H(\xi-\xi_0-(\eta-1)\sqrt{\Xi})$$
$$\times\left[\frac{(\Xi+\xi-\xi_0-(\eta-1)\sqrt{\Xi})\gamma-1}{(\xi_1-\xi_0)\gamma^2}\,e^{-\gamma(\eta-1)\sqrt{\Xi}}\right.$$
$$\left.-\frac{\Xi\gamma-1}{(\xi_1-\xi_0)\gamma^2}\,e^{-\gamma(\xi-\xi_0)}\right]$$
$$-H(\xi-\xi_1-(\eta-1)\sqrt{\Xi})$$
$$\times\left[\frac{(\Xi+\xi-\xi_0-(\eta-1)\sqrt{\Xi})\gamma-1}{(\xi_1-\xi_0)\gamma^2}\,e^{-\gamma(\eta-1)\sqrt{\Xi}}\right.$$
$$\left.+\frac{(\Xi\gamma-1)[(\xi_1-\xi_0)\gamma-1]}{(\xi_1-\xi_0)\gamma^2}\,e^{-\gamma(\xi-\xi_1)}\right]. \qquad (39)$$

The dimensionless temperature field can be evaluated in the case of a heat flux pulse with the shape of either a square wave or a triangular wave by employing equations (30), (31) and (38) or equations (30), (31) and (39), respectively.

In the case of parabolic heat conduction Ξ is zero, so that equation (38) can be rewritten as

$$\psi(\gamma,\eta,\xi) = H(\xi-\xi_0)\frac{1-e^{-\gamma(\xi-\xi_0)}}{\gamma}$$
$$-H(\xi-\xi_1)\frac{1-e^{-\gamma(\xi-\xi_1)}}{\gamma} \qquad (40)$$

while equation (39) can be rewritten as

$$\psi(\gamma,\eta,\xi) =$$
$$H(\xi-\xi_0)\frac{(\xi-\xi_0)\gamma-1+e^{-\gamma(\xi-\xi_0)}}{(\xi_1-\xi_0)\gamma^2}$$
$$-H(\xi-\xi_1)$$
$$\times\frac{(\xi-\xi_0)\gamma-1-[(\xi_1-\xi_0)\gamma-1]\,e^{-\gamma(\xi-\xi_1)}}{(\xi_1-\xi_0)\gamma^2}. \qquad (41)$$

In the case $\Xi = 0$, equation (30) yields $\rho(\gamma) = \sqrt{\gamma}$. Therefore, by employing equations (30), (31) and (40) or equations (30), (31) and (41), the dimensionless temperature field can be evaluated for parabolic heat conduction in the case of a heat flux pulse with the shape of a square wave or of a triangular wave, respectively.

In Carslaw and Jaeger [16], the parabolic and nonstationary heat conduction in the region $r > r_0$ with a prescribed constant and uniform heat flux at the surface $r = r_0$ has been studied. The analytical expression of the temperature field obtained in [16] must coincide, in the limit of a vanishing thermal relaxation time, with the solution of the hyperbolic heat conduction equation obtained in this paper in the case of a square wave pulse with $\xi_0 \to 0$ and $\xi_1 \to +\infty$. Indeed, by taking the limits $\xi_0 \to 0$ and $\xi_1 \to +\infty$, equation (40) yields

$$\psi(\gamma,\eta,\xi) = H(\xi)\frac{1-e^{-\gamma\xi}}{\gamma}. \qquad (42)$$

By substituting equation (42) in equation (31) and by employing the equation $\rho(\gamma) = \sqrt{\gamma}$, one obtains

$$\vartheta(\eta,\xi) = \frac{1}{\pi}H(\xi)$$
$$\times\int_0^{+\infty}\frac{J_1(\sqrt{\gamma})Y_0(\eta\sqrt{\gamma})-Y_1(\sqrt{\gamma})J_0(\eta\sqrt{\gamma})}{J_1(\sqrt{\gamma})^2+Y_1(\sqrt{\gamma})^2}$$
$$\times\frac{1-e^{-\gamma\xi}}{\gamma^{3/2}}\,d\gamma. \qquad (43)$$

It is easily proved that equation (43) can be rewritten as

$$\vartheta(\eta,\xi) = \frac{2}{\pi}H(\xi)$$
$$\times\int_0^{+\infty}\frac{J_1(u)Y_0(\eta u)-Y_1(u)J_0(\eta u)}{J_1(u)^2+Y_1(u)^2}$$
$$\times\frac{1-e^{-u^2\xi}}{u^2}\,du \qquad (44)$$

where $u = \sqrt{\gamma}$. Equations (9) and (44) yield the same temperature field as that obtained in Carslaw and Jaeger [16] in the case of constant and uniform heat flux at $r = r_0$.

In Figs. 3–10, plots of the dimensionless temperature ϑ vs the dimensionless time ξ for $\eta = 1$ and $\eta = 10$ and for $\Xi = 1$ and $\Xi = 8$, either in the case of a square wave pulse or in the case of a triangular wave pulse are reported. These plots show that hyperbolic heat conduction produces discontinuous variations of temperature with time which are absent if parabolic heat conduction is considered. This feature has been

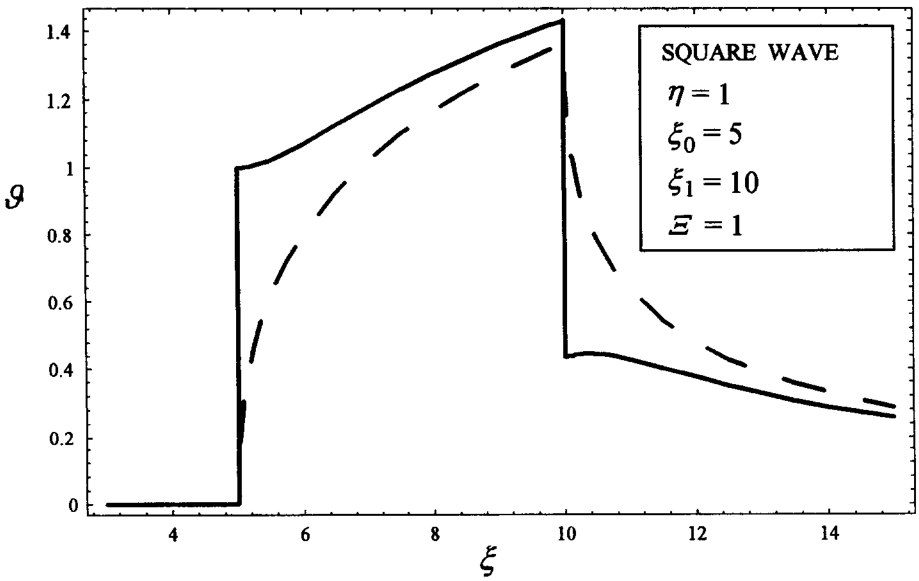

Fig. 3. Plots of ϑ vs ξ for $\eta = 1$ for hyperbolic conduction with $\Xi = 1$ (solid line) and for parabolic conduction (dashed line) in the case of a square wave pulse.

pointed out in other papers on thermal waves dealing with boundary conditions of prescribed wall heat fluxes which vary discontinuously with time [9–12]. Moreover, comparisons between Figs. 3 and 5, between Figs. 4 and 6, between Figs. 7 and 9 or between Figs. 8 and 10 show that, at the position $r = 10r_0$, a time delay in the arrival of the thermal signal due to the finite speed of propagation is present: such a time delay is absent in the case of parabolic heat conduction. Another relevant feature of hyperbolic heat conduction is shown in Figs. 4 and 8: after the heat flux is switched off, at $r = r_0$ the temperature falls below (if $q_0 > 0$) or jumps above (if $q_0 < 0$) its initial value T_0. If hyperbolic heat conduction were a theory based on the local equilibrium hypothesis as parabolic heat conduction is, this behaviour would represent a violation of Clausius' statement of the second law. Indeed, it has been pointed out in the

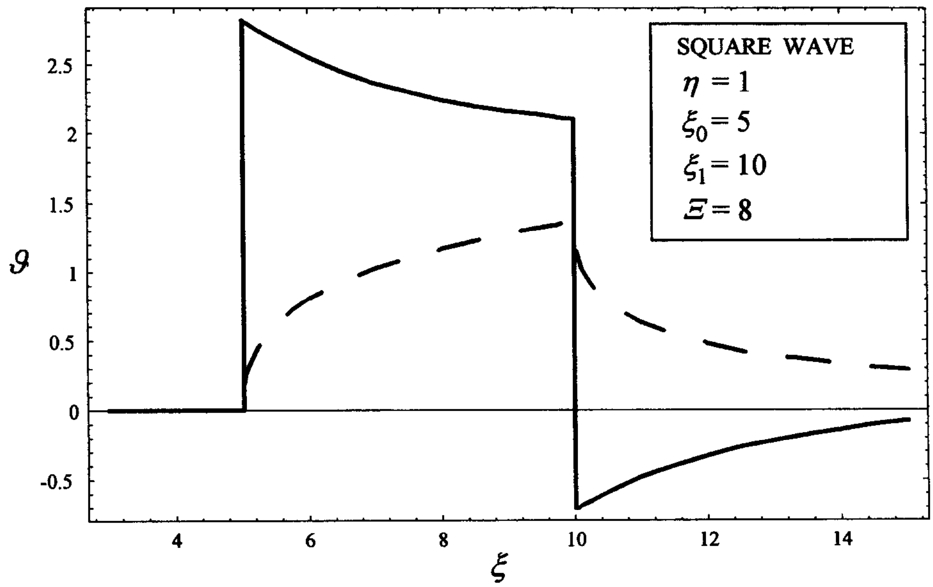

Fig. 4. Plots of ϑ vs ξ for $\eta = 1$ for hyperbolic conduction with $\Xi = 8$ (solid line) and for parabolic conduction (dashed line) in the case of a square wave pulse.

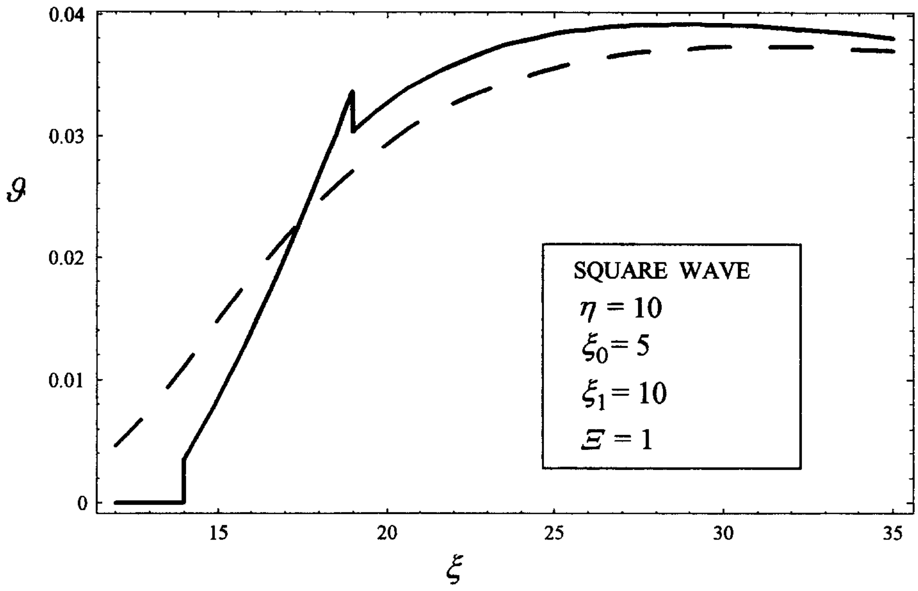

Fig. 5. Plots of ϑ vs ξ for $\eta = 10$ for hyperbolic conduction with $\Xi = 1$ (solid line) and for parabolic conduction (dashed line) in the case of a square wave pulse.

literature that hyperbolic heat conduction is in contrast with the local equilibrium hypothesis [2], so that no violation of the principles of thermodynamics occurs. The undercooling/overheating of the solid material after the switching off of the heat flux is present also for $r > r_0$, but becomes less relevant as r increases and at a sufficient distance from the surface $r = r_0$ this effect disappears. In fact, for $r = 10r_0$, Figs. 6 and 10 show that no undercooling/overheating occurs. Moreover, Figs. 3, 5, 7 and 9 shows that the undercooling/overheating effect does not occur for low values of Ξ, as for instance $\Xi = 1$.

CONCLUSIONS

Hyperbolic heat conduction in an infinite solid medium internally bounded by a cylindrical surface has been analysed. On this surface a uniform and time-varying heat flux has been prescribed. It has been assumed that, in the initial state, the solid has a steady and uniform temperature distribution. The heat conduction equation together with its boundary and initial conditions have been written in a dimensionless form. By employing the Laplace transform technique, an analytical solution has been found for an arbitrary

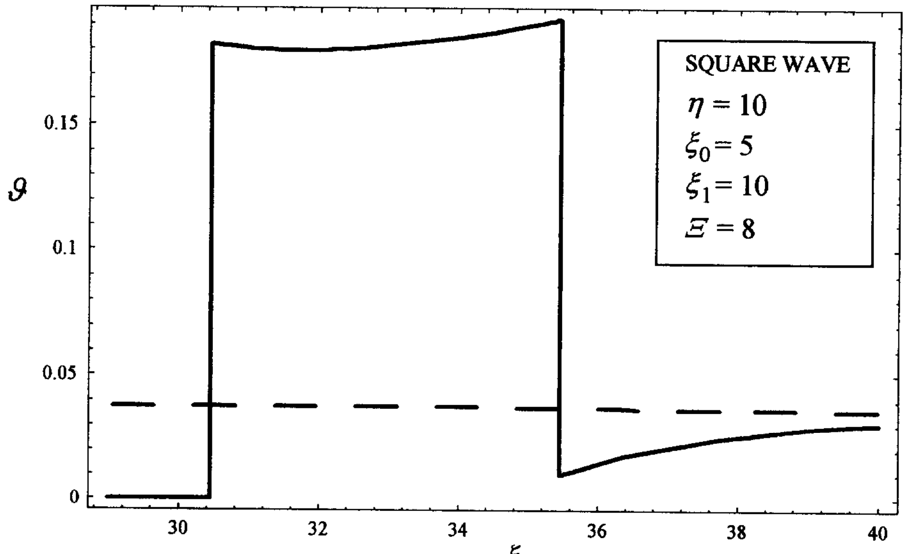

Fig. 6. Plots of ϑ vs ξ for $\eta = 10$ for hyperbolic conduction with $\Xi = 8$ (solid line) and for parabolic conduction (dashed line) in the case of a square wave pulse.

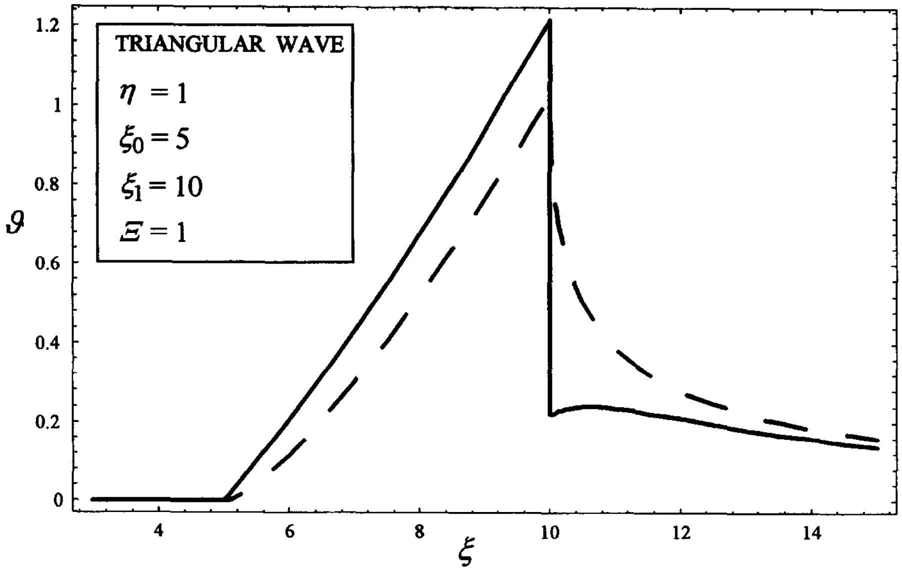

Fig. 7. Plots of ϑ vs ξ for $\eta = 1$ for hyperbolic conduction with $\Xi = 1$ (solid line) and for parabolic conduction (dashed line) in the case of a triangular wave pulse.

time variation of the heat flux at $r = r_0$. In the case of a heat flux which behaves like a pulse, plots of the dimensionless temperature ϑ vs the dimensionless time ξ have been obtained for a square wave pulse and for a triangular wave pulse. These plots reveal two important features of hyperbolic heat conduction which are not presented by parabolic heat condition: (a) both for a square wave pulse and for a triangular wave pulse, the discontinuities in the time-variation of the heat flux produce discontinuities in the time-variation of the temperature field; (b) after the switching-off of the heat flux, for a sufficiently high value of the thermal relaxation time, both for a square wave pulse and for a triangular wave pulse, the temperature at $r = r_0$ falls below (if $q_0 > 0$) or jumps above (if $q_0 < 0$) its initial value.

Feature (b) is not in contrast with the principles of thermodynamics, but merely reveals a conflict between the theory of hyperbolic heat conduction and the hypothesis of local equilibrium. Indeed, among the cases which have been considered in the plots, this apparent violation of Clausius' statement of the second law occurs only when the thermal relaxation time is greater than the duration of the pulse, i.e. in highly non-stationary cases.

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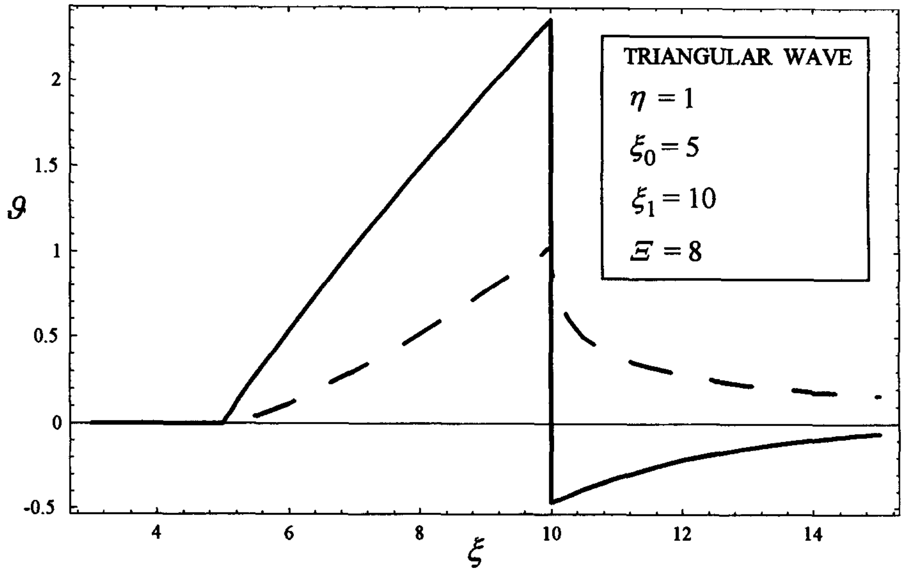

Fig. 8. Plots of ϑ vs ξ for $\eta = 1$ for hyperbolic conduction with $\Xi = 8$ (solid line) and for parabolic conduction (dashed line) in the case of a triangular wave pulse.

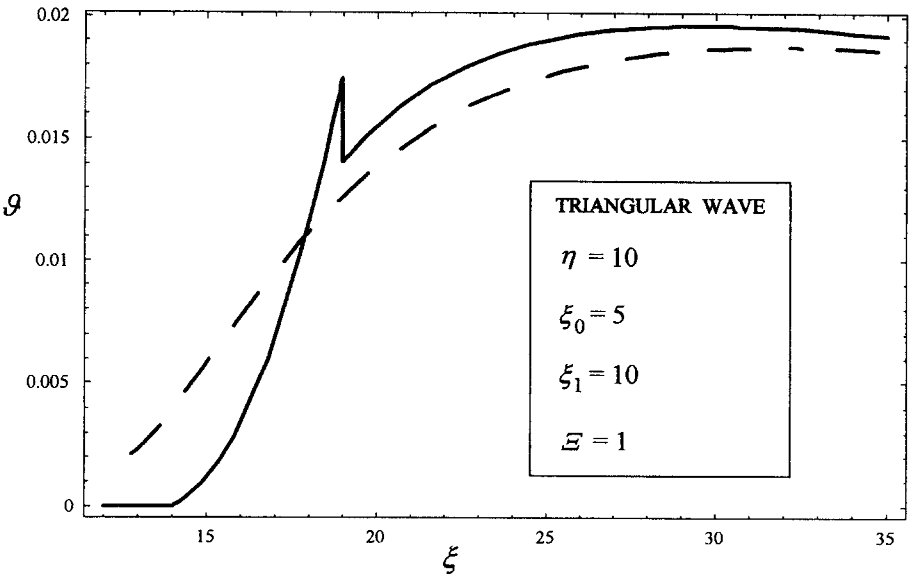

Fig. 9. Plots of ϑ vs ξ for $\eta = 10$ for hyperbolic conduction with $\Xi = 1$ (solid line) and for parabolic conduction (dashed line) in the case of a triangular wave pulse.

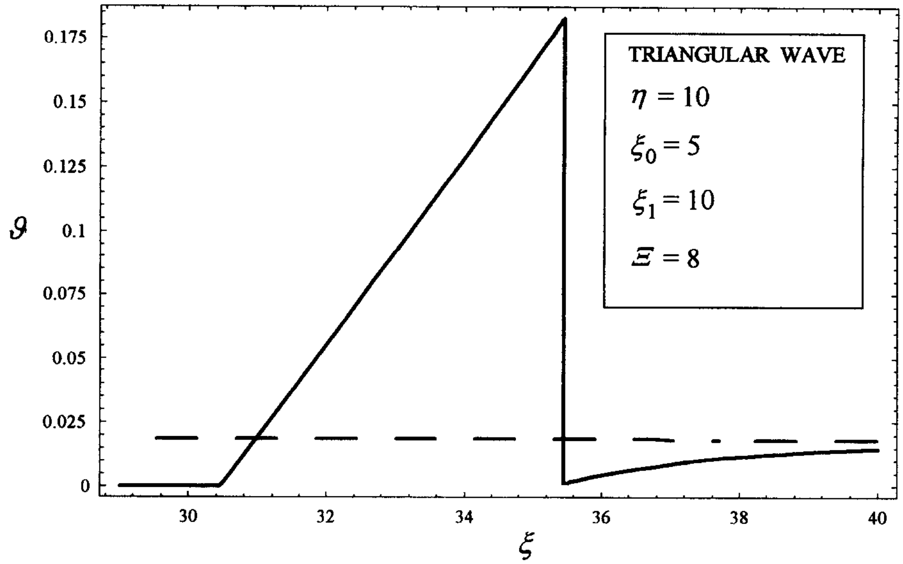

Fig. 10. Plots of ϑ vs ξ for $\eta = 10$ for hyperbolic conduction with $\Xi = 8$ (solid line) and for parabolic conduction (dashed line) in the case of a triangular wave pulse.

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APPENDIX

Function $a(\eta, \xi)$ can be evaluated by employing the inversion formula (28) and the closed contour Σ represented in Fig. 1. The integrand which appears in equation (28) has two branch points: $s = 0$ and $s = -1/\Xi$, so that a branch cut is given by $\mathrm{Im}(s) = 0$ and $\mathrm{Re}(s) < 0$. Equation (28) can be rewritten as

$$a(\eta, \xi) = \frac{1}{2\pi i} \lim_{R \to +\infty} \lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to 0} \times \left[\oint_{\Sigma} e^{s\xi} \tilde{a}(\eta, s)\, ds - \int_{\Gamma_R} e^{s\xi} \tilde{a}(\eta, s)\, ds - \int_{\Gamma_{\varepsilon_1}} e^{s\xi} \tilde{a}(\eta, s)\, ds - \int_{\Gamma_{\varepsilon_2}} e^{s\xi} \tilde{a}(\eta, s)\, ds - \int_{AB} e^{s\xi} \tilde{a}(\eta, s)\, ds - \int_{CD} e^{s\xi} \tilde{a}(\eta, s)\, ds - \int_{EF} e^{s\xi} \tilde{a}(\eta, s)\, ds - \int_{GH} e^{s\xi} \tilde{a}(\eta, s)\, ds \right] \quad \text{(A1)}$$

where

$$\tilde{a}(\eta, s) = \frac{K_0(\eta\sqrt{(s + \Xi s^2)})\, e^{(\eta - 1)s\sqrt{\Xi}}}{K_1(\sqrt{(s + \Xi s^2)})\sqrt{(s + \Xi s^2)}}. \quad \text{(A2)}$$

It is well known that there is no zero of function K_1 [17], so that there is no pole of $\tilde{a}(\eta, s)$ within the region bounded by Σ. Therefore, on account of the residue theorem [15], the contour integral on Σ which appears in equation (A3) is zero. By employing the asymptotic expression of K_n, which holds for large values of its argument [14],

$$K_n(z) \approx \frac{e^{-z}}{\sqrt{(2\pi z)}} \quad \text{(A3)}$$

it is easily proved that, in equation (A1), the integral on the semicircular path Γ_R vanishes in the limit $R \to +\infty$. Moreover, on account of the expressions of K_0 and K_1 for small values of their arguments [14],

$$K_0(z) \approx -\ln\left(\frac{z}{2}\right) \quad \text{(A4)}$$

$$K_1(z) \approx \frac{z}{2} \ln\left(\frac{z}{2}\right) \quad \text{(A5)}$$

it can be easily verified that, in equation (A1), the integrals on the circular paths Γ_{ε_1} and Γ_{ε_2} vanish in the limits of $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, respectively.

As a consequence of equation (A2), one obtains

$$-\frac{1}{2\pi i} \lim_{R \to +\infty} \lim_{\varepsilon_1 \to 0} \left[\int_{AB} e^{s\xi} \tilde{a}(\eta, s)\, ds + \int_{GH} e^{s\xi} \tilde{a}(\eta, s)\, ds \right] = \frac{1}{\pi} \mathrm{Re}\left\{ \int_{1/\Xi}^{+\infty} \frac{K_0(\eta\rho(y)\, e^{i\pi})\, e^{-y(\xi + (\eta - 1)\sqrt{\Xi})}}{i K_1(\rho(y)\, e^{i\pi})\rho(y)}\, dy \right\}. \quad \text{(A6)}$$

By employing the identity [18]

$$K_n(e^{i\pi} z) = (-1)^n K_n(z) - i\pi I_n(z) \quad \text{(A7)}$$

equation (A6) can be rewritten as

$$-\frac{1}{2\pi i} \lim_{R \to +\infty} \lim_{\varepsilon_1 \to 0} \times \left[\int_{AB} e^{s\xi} \tilde{a}(\eta, s)\, ds + \int_{GH} e^{s\xi} \tilde{a}(\eta, s)\, ds \right] = \int_{1/\Xi}^{+\infty} \frac{I_1(\rho(y))K_0(\rho(y)\eta) + K_1(\rho(y))I_0(\rho(y)\eta)}{\rho(y)[\pi^2 I_1(\rho(y))^2 + K_1(\rho(y))^2]} \times e^{-y[\xi + (\eta - 1)\sqrt{\Xi}]}\, dy. \quad \text{(A8)}$$

Moreover, by employing equation (A2), one obtains

$$-\frac{1}{2\pi i} \lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to 0} \left[\int_{CD} e^{s\xi} \tilde{a}(\eta, s)\, ds + \int_{EF} e^{s\xi} \tilde{a}(\eta, s)\, ds \right] = \frac{1}{\pi} \mathrm{Re}\left\{ \int_0^{1/\Xi} \frac{K_0(\eta\rho(y)\, e^{i\pi/2})\, e^{-y(\xi + (\eta - 1)\sqrt{\Xi})}}{K_1(\rho(y)\, e^{i\pi/2})\rho(y)}\, dy \right\}. \quad \text{(A9)}$$

On account of the identity [18]

$$K_n(e^{i\pi/2} z) = \frac{\pi}{2}(-i)^{n+1}[J_n(z) - i Y_n(z)] \quad \text{(A10)}$$

equation (A9) can be rewritten as

$$-\frac{1}{2\pi i} \lim_{\varepsilon_1 \to 0} \lim_{\varepsilon_2 \to 0} \times \left[\int_{CD} e^{s\xi} \tilde{a}(\eta, s)\, ds + \int_{EF} e^{s\xi} \tilde{a}(\eta, s)\, ds \right] = \frac{1}{\pi} \int_0^{1/\Xi} \frac{J_1(\rho(y))Y_0(\rho(y)\eta) - Y_1(\rho(y))J_0(\rho(y)\eta)}{\rho(y)[J_1(\rho(y))^2 + Y_1(\rho(y))^2]} \times e^{-y[\xi + (\eta - 1)\sqrt{\Xi}]}\, dy. \quad \text{(A11)}$$

As a consequence of equations (A1), (A8) and (A11), equation (29) holds.